ALGEBRAIC FUNCTIONS OF NORMAL OPERATORS*

BY

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ABSTRACT

The solutions of $p(T) = S$, where S is normal and p a polynomial, are described.

Let p be a polynomial of degree n $p(z) = z^n + a_1 z^{n-1} + \cdots + a_0$. Let $u_1 \cdots u_k$ be the different solutions of $p'(z) = 0$ $(k \leqq n-1)$ and put $v_i = p(u_i)$. If $w \neq v_i$ then $p(z) = w$ has n different solutions $z_1(w), \cdots, z_n(w)$ and we shall assume

The functions $z_i(w)$ are measurable.

It is possible to construct such measurable roots see [2, Lemma 3.1]. Let S be a normal operator on the Hilbert space H and $E(\cdot)$ the spectral measure of S.

THEOREM. *Let $v_i \notin \sigma(S)$ $i = 1, \cdots, k$. The bounded operator T satisfies $p(T) = S$ if and only if*

$$(1) \qquad T = \sum_{i=1}^{n} F_i \int z_i(w) E(dw) \quad F_i^2 = F_i, \; F_i F_j = 0 \; i \neq j, \; \sum_{i=1}^{n} F_i = I$$

and $F_i S = S F_i$.

Proof. For any $w \in \sigma(S)$

$$p(z) - w = (z - z_1(w)) \cdots (z - z_n(w))$$

hence

$$p(T) - w = (T - z_1(w)) \cdots (T - z_n(w)).$$

Now T commutes with $S = p(T)$ hence with $E(\cdot)$. Integrate the last equation, using the multiplicativity of $E(\cdot)$ to derive

(a) $0 = p(T) - S = (T - \int z_1(w) E(dw)) \cdots (T - \int z_n(w) E(dw))$.

Also

$$p'(z) = \sum_{j=1} \prod_{i \neq j} (z - z_i(w)) \text{ hence } p'(T) = \sum_{j=1} \prod_{i \neq j} (T - z_i(w))$$

again integrate

(b) $p'(T) = \sum_{j=1}^{n} \prod_{i \neq j} (T - \int z_i(w) E(dw))$.

Now $p'(T)^{-1}$ exists since $v_i \notin \sigma(S)$. Put $F_j = p'(T)^{-1} \prod_{i \neq j} (T - \int z_i(w) E(dw))$.

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Note that $F_iF_j = 0$ by (a) and $\sum F_i = I$ by (b). Thus $F_i^2 = F_i \sum_{j=1}^n F_j = F_i$. Finally $(T - \int z_i(w)E(dw))F_i = 0$ by (a) hence $T = \sum TF_i = \sum F_i \int z_i(w)E(dw)$. It is clear that any operator of the form (1) solves $p(T) = S$.

The same argument works for S a scalar operator in the sense of N. Dunford. A different representation can be obtained by using Theorem 1 of [5]. Thus let $A > 0$ satisfy: $A^{-1}F_iA$ and $A^{-1}E(\cdot)A$ are self adjoint projections. Now

$$A^{-1}E(\alpha)A = (A^{-1}E(\alpha)A)^* = AE(\alpha)A^{-1}$$

or A^2 commutes with $E(\alpha)$ thus the spectral measure of A^2 commutes with $E(\alpha)$ and so does $A(A > 0)$.

Put $\tilde{F}_i = A^{-1}F_iA$ to replace (1) by

$$T = A \sum_{i=1}^{n} \int z_i(w)E(dw)\tilde{F}_iA^{-1} \tag{2}$$

where $\tilde{F}_i$ *are self adjoint projections,* $\tilde{F}_i\tilde{F}_j = 0$ $i \neq j$, $\sum \tilde{F}_i = I$, $S\tilde{F}_i = \tilde{F}_iS$ and $SA = AS$.

Note that $\tilde{F}_i$ does not have to commute with A.

REMARK. The form (1) was obtained by J. G. Stampfli [3], for roots of scalar operators ($T^n = S$ where S is scalar). The form (2) was obtained by L. Wallen [4], for roots of unitary operators ($T^n = S$ where S is unitary).

If the multiplicity of S is greater than 1 it is easy to construct examples with $F_i \neq F_i^*$ (or $A\tilde{F}_i \neq \tilde{F}_iA$).

Let us consider the case where $\sigma(S)$ contains v_i.

Let α_m be disjoint Borel sets whose union is $\sigma(S) - \langle v_1, \cdots v_k \rangle$. Then if $p(T) = S$ we get

$$p(TE(\alpha_m)) = SE(\alpha_m)$$

$$p(TE(\langle v_i \rangle)) = v_iE(\langle v_i \rangle).$$

Thus by the Theorem on $E(\sigma(S) - \langle v_1 \cdots v_k \rangle)H$

$$T = \sum_{m=1}^{\infty} \sum_{i=1}^{n} \int z_i(w)E(dw)F_{i,m}E(\alpha_m)$$

but $\sum_m F_{i,m}E(\alpha_m)$ is not necessarily bounded, or

$$T = \sum_{m=1}^{\infty} \sum_{i=1}^{n} \int z_i(w)E(dw)A_m\tilde{F}_{i,m}A_m^{-1}E(\alpha_m)$$

where $\sum A_mE(\alpha_m)$ is not necessarily bounded. Finally as on $E(\langle v_i \rangle)H$ the solutions of $p(T) = v_i$ are probably known, let us describe them for completeness sake.

If $p(u_1) = \cdots = p(u) = v_i$ then $\sigma(T) \subset \langle u_1 \cdots u \rangle$

$$T = \sum u_jG_j + N = T_1 + N$$

where G_j is the projection obtained by integrating $(\lambda - T)^{-1}$ on a small circle around u_j and N is a quasi nilpotent that commutes with them, also by [1 Theorem 9]

$$v_i I = p(T) = p(T_1) + Np'(T_1) + \cdots + \frac{N^n}{n!} p^{(n)}(T_1)$$

or

$$N\left(p'(T_1) + \cdots + \frac{N^{n-1}}{n!} p^{(n)}(T_1)\right) = 0.$$

nus if $p'(u_j) \neq 0$ then on $G_i H$ the operator $p'(T_1) + \cdots + \frac{N^{n-1}}{n!} p^{(n)}(T_1)$ has an inverse or $NG_i = 0$. If $p'(w_i) = 0$ and $p''(w_i) \neq 0$ then on G_i

$$N^2 \left[\frac{p''(T_1)}{2} + \cdots + \frac{N^{n-2}}{n!} p^{(n)}(T_1)\right] = 0$$

and again $N^2 G_i = 0$. Finally a zero of $p(z) - v$ is of order not greater than $n - 1$ and thus $N^n = 0$.

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